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Special Theory of Relativity

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Introduction

- Basic concept in NM
- Three laws of motion
- Frame of References
- Inertial System (Frame)
- Non – Inertial System
- Event

In an inertial frame a body, not under the influence of any force, moves in a straight line with constant speed.

Since no force acts on a body,

$$\frac{d\bar{p}}{dt} = \bar{F} = m\bar{a} = 0$$

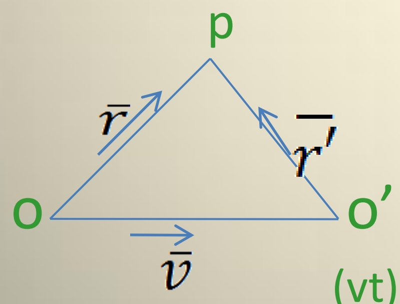
ie $\frac{d^2\bar{r}}{dt^2} = 0$ where $\bar{r}$ is the p. v. of a moving body at a time t.

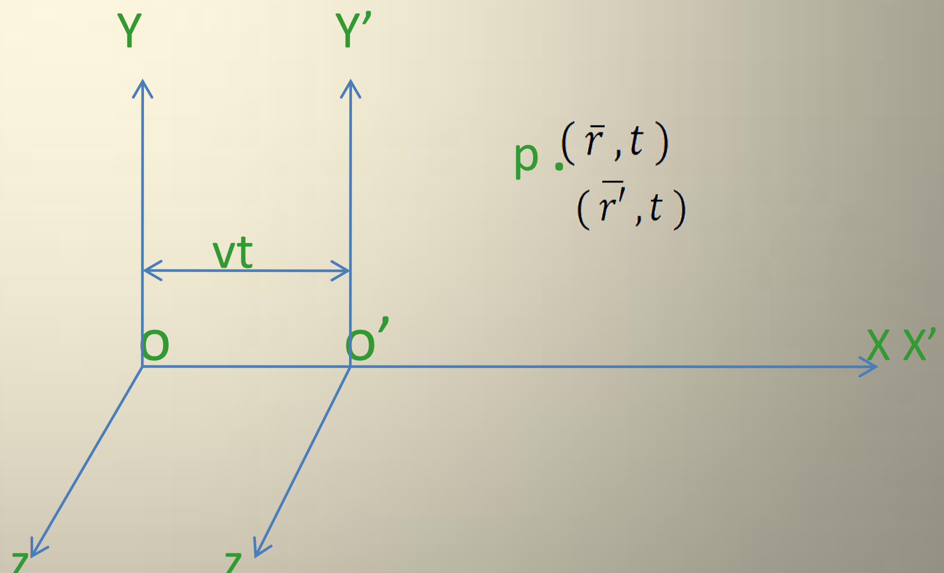
$$\frac{d^2\bar{r}}{dt^2} = 0 \Rightarrow \frac{d\bar{v}}{dt} = 0 \Rightarrow \bar{v} = const.$$

$\bar{v} = \frac{d\bar{r}}{dt} = a \Rightarrow \bar{r} = at + b$ which is a straight line.

Galilean Transformation (GT)

- Consider two inertial frames S and S'
- S' is moving with uniform speed **v** w. r. to S.
- O & O' Observers at origins of S & S' observing the same event at any point p.





By Triangle law of addition,

$$\bar{r} = \overline{oo'} + \overline{r'}$$

$$\Rightarrow \overline{r'} = \bar{r} - \bar{v}\,t \quad \text{where} \quad \overline{oo'} = \bar{v}t \text{ …….(1)}$$

The time can be defined independently.

$\therefore$ t’ = t ……………………………………………..(2)

These equations are Galilean Transformations.

If observer o’ moves along xx’ direction, then

$$\bar{v} = (v, 0, 0)$$

$\therefore$ (1) & (2) $\Rightarrow$ (x’, y’ z’) = (x, y, z) – (v, 0, 0) t

$$\Rightarrow \quad x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

$\therefore$ Galilean Inverse Transformations are

$$\bar{r} = \overline{r'} + \bar{v}\,t \quad , \qquad t = t'$$

$$x = x' + vt\,, \;\; y = y'\,, \;\; z = z'\,, \; t = t'.$$

Principle of Relativity

Galilean Transformations are

$$\overline{r'} = \bar{r} - \bar{v}\, t, \qquad t' = t$$

Differentiating w. r. to t.

$$\frac{d\overline{r'}}{dt} = \frac{d\bar{r}}{dt} - \bar{v} \qquad \text{ie} \qquad \overline{u'} = \bar{u} - \bar{v}$$

Where $\bar{u}$ is the velocity of a particle in S.

$$\Rightarrow \frac{d\overline{u'}}{dt} = \frac{d\bar{u}}{dt} \qquad \Rightarrow \overline{a'} = \bar{a}$$

Where $\bar{a}$ is the acceleration of the particle in S.
In NM mass does not depend upon the motion of a particle ie. m' = m

$$\therefore \overline{F'} = m'\overline{a'} = m\bar{a} = \bar{F}$$

Ex. Show that Newton's kinematical equations of motion are invariant under GT

Sol. : Let a particle is moving in an inertial frame S. Its Newton's kinematical equations are

$$V = U + A\,t \,\ldots\ldots.(1)$$

$$V^2 = U^2 + 2AS\ldots\ldots\ldots(2)$$

$$S = U\,t + \tfrac{1}{2} A\,t^2 \;\ldots\ldots\ldots..(3)$$

Let S' is moving with velocity v along xx' axis.

$\therefore \quad V' = V - v \quad$ and $\quad U' = U - v \;\ldots\ldots\ldots\ldots\ldots\ldots..(4)$

$\Rightarrow V' - U' = V - U = A\,t \quad$ from (4 & 1)

$\Rightarrow V' = U' + A\,t' \quad$ where $t = t' \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(5)$

By GT $S' = S - v\,t$

$= U\,t + \frac{1}{2} A\,t^2 - v\,t$

$= (U - v)\,t + \frac{1}{2} A\,t^2$

$\therefore$ $S' = U'\,t + \frac{1}{2} A\,t'^2$ where $t' = t$ ....(6)

Now, $V'^2 - U'^2 = (V - v)^2 - (U - v)^2$

$= V^2 - U^2 - 2v\,(V - U)$

$= 2\,A\,S - 2v\,A\,t$

$= 2\,A\,(S - v\,t)$

$= 2\,A\,S'$

$\Rightarrow$ $V'^2 = U'^2 + 2\,A\,S'$ ....(7)

$\therefore$ Kinematical equations of motion are invariant under GT.

Maxwell’s Electromagnetic Theory

Maxwell’s equations in vacuum are

Curl E = $-\frac{1}{c}\frac{\partial H}{\partial t}$, div H = 0 , Curl H = $\frac{1}{c}\frac{\partial E}{\partial t}$, div E = 0

These equations have non-zero solutions and leads to the wave equation in the form

$$\nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$$

If these equations remain invariant under GT then we say that PR is valid for electromagnetic theory. But mathematical calculations gives negative result.

Ex. Prove that the wave equation $\nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$ do not remain invariant under GT

Sol^n : Consider two inertial frames, S & S'.

GT are $x' = x - vt$, $y' = y$, $z' = z$, $t' = t$

From the Calculus, we have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x} + \frac{\partial f}{\partial z'}\frac{\partial z'}{\partial z} + \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial t}$$

$$= \frac{\partial f}{\partial x'} .1 + 0 + 0 + 0 \qquad \text{by GT}$$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'} \Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x'^2} \;\&\; \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x'^2}$$

Similarly we have $\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial y'^2} \;\&\; \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial z'^2}$

Also we have,

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial t} + \frac{\partial f}{\partial z'}\frac{\partial z'}{\partial t} + \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial t}$$

By GT,

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x'}(-\mathrm{v}) + 0 + 0 + \frac{\partial f}{\partial t'}.1$$

$$\therefore \frac{\partial f}{\partial t} = -\mathrm{v}\frac{\partial f}{\partial x'} + \frac{\partial f}{\partial t'}$$

$$\text{or } \frac{\partial}{\partial t} = -\mathrm{v}\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \left(-v\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\right)\left(-v\frac{\partial f}{\partial x'} + \frac{\partial f}{\partial t'}\right)$$

$$\frac{\partial^2 f}{\partial t^2} = v^2\frac{\partial^2 f}{\partial x'^2} - v\frac{\partial^2 f}{\partial x'\partial t'} - v\frac{\partial^2 f}{\partial t'\partial x'} + \frac{\partial^2 f}{\partial t'^2}$$

$$\therefore\ \nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{1}{c^2}\left(v^2\frac{\partial^2 f}{\partial x'^2} - 2v\frac{\partial^2 f}{\partial x'\partial t'} + \frac{\partial^2 f}{\partial t'^2}\right)$$

$$\neq \nabla'^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t'^2}$$

$$\therefore\ \nabla^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} \neq \nabla'^2 f - \frac{1}{c^2}\frac{\partial^2 f}{\partial t'^2}$$

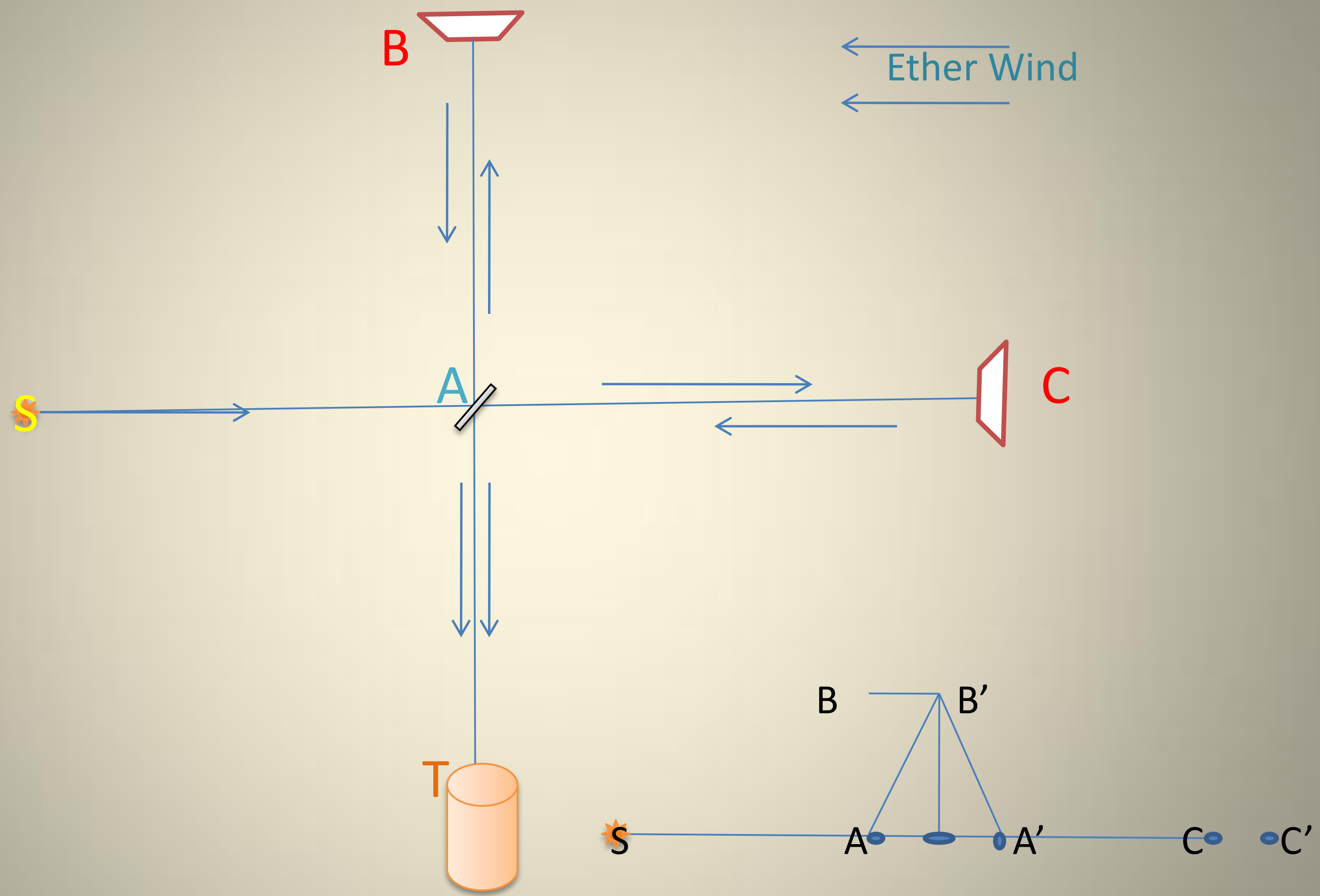
Otherwise $v = 0$

Therefore Maxwell's wave equation do not remain invariant under GT.

ie Maxwell's equations do not obey PR.

ie electromagnetic effects are different in different frames.

Michelson – Morley Experiment (MM Expt.)



Let

$c \equiv$ velocity of light

$V \equiv$ velocity of the earth / apparatus.

$t_1 \equiv$ time for a beam to travel from A to B & back.

$t_2 \equiv$ time for a beam to travel from A to C & back.

Let, $AB = l_1$ & $AC = l_{2.}$

From the fig. ie. Triangle AOB'

$$AB'^2 = OB'^2 + OA^2$$

$AB'^2 = AB^2 + BB'^2$ [where OA=BB' & OB'=AB]

$$\therefore \; c^2 t^2 = l_1{}^2 + v^2 t^2$$

where $AB' = ct$, $BB' = vt$, $AB = l_1$

$$\therefore \; t = \frac{l_1}{\sqrt{c^2 - v^2}}$$

$$\therefore \quad t_1 = \frac{2l_1}{\sqrt{c^2 - v^2}} =$$ time for a beam to travel from A to B & back

$$\text{ie.}\, t_1 = \frac{2l_1}{c\sqrt{1 - {v^2}/{c^2}}}$$

$$\text{ie.}\, t_1 = \frac{2l_1}{c\sqrt{1 - \beta^2}}$$ where $\beta = {v}/{c}$

For the path AC the relative velocity of the beam is $c - v$ & $c + v$.

Let,

t_2 ≡ time for the beam to travel from A to C & back.

$$\therefore \quad t_2 = \frac{l_2}{c - v} + \frac{l_2}{c + v}$$

$$t_2 = l_2\left[\frac{c + v + c - v}{c^2 - v^2}\right]$$

$$t_2 = \frac{2cl_2}{c^2 - v^2}$$

$$\therefore \quad t_2 = \frac{2l_2}{c(1 - \beta^2)}$$ where $\beta = {}^{v}/_{c}$

Time Difference, $\Delta = t_2 - t_1$

$$\Delta = \frac{2l_2}{c(1-\beta^2)} - \frac{2l_1}{c\,(1-\beta^2)^{1/2}}$$

$$\Delta = \frac{2l_2}{c}(1-\beta^2)^{-1} - \frac{2l_1}{c}(1-\beta^2)^{-1/2}$$

Since, v<<c,

$\beta = {}^{v}/_{c} << 1$

$$\Delta = \frac{2l_2}{c}(1+\beta^2) - \frac{2l_1}{c}\left(1+\frac{\beta^2}{2}\right)$$

$$\Delta = \frac{2}{c}\left[\, l_2 - l_1 + l_2\beta^2 - l_1\frac{\beta^2}{2}\right]$$

$$\Delta = \frac{2}{c}(l_2 - l_1) + \frac{2}{c}\left(l_2 - \frac{l_1}{2}\right)\beta^2$$

$$\Delta = \frac{2}{c}(l_2 - l_1) + \left(\frac{2l_2 - l_1}{c}\right)\beta^2 \quad \text{............(1)}$$

Now whole apparatus is turned through an angle 90^0 so that Ether wind is perpendicular to AC. In this case,
Time Difference,

$$\Delta' = t'_2 - t'_1$$

$$\Delta' = \frac{2}{c}(l_2 - l_1) + \left(\frac{l_2 - 2l_1}{c}\right)\beta^2 \qquad \text{.........(2)}$$

The relation between N, the no. interference fringes and change of time difference is,

$$N = \frac{c(\Delta - \Delta')}{\lambda}$$

Where λ is the wavelength of light.
From equations (1) & (2) we have

$$N = \frac{c}{\lambda}\left\{\frac{2}{c}(l_2 - l_1) + \left(\frac{2l_2 - l_1}{c}\right)\beta^2 - \left[\frac{2}{c}(l_2 - l_1) + \left(\frac{l_2 - 2l_1}{c}\right)\beta^2\right]\right\}$$

$$N = \frac{c}{\lambda}\left[\frac{\beta^2}{c}\left(2l_2 - l_1 - l_2 + 2l_1\right)\right.$$

$$N = \frac{\beta^2}{\lambda}\left(l_1 + l_2\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

In MM Expt. $l_1 + l_2 = 22$ m. Taking $\lambda = 5.5 \text{ x } 10^{-7}$

and $\beta = \frac{v}{c} = 10^{-4}$

We get N = 0.4

ie. Fringe shift exists.

But Michelson and Morley did not find any fringe shift ie. N = 0.

This result implies that $\boxed{\beta = 0 \text{ ie. } v = 0}$

Fitzgerald and Lorentz contraction hypothesis

There is a real contraction along the direction of motion by the factor,

$$\left(1-\frac{v^2}{c^2}\right)^{1/2} \text{ or } (1-\beta^2)^{1/2}$$

This hypothesis implies that N=0.

If l and l^0 are the lengths of the body in motion and the body at rest then

$$l = l^0\sqrt{1-\beta^2}$$

Einstein special theory of relativity

Every body will agree that the laws of physics should not change from one frame to another frame.ie. All frames are equivalent (PR). But Maxwell's equations of electromagnetic theory do not remain invariant under GT.

Alternatives

- PR applies to mechanics and not to electromagnetic theory
- PR applies to NM & electromagnetic theory but electro-magnetic theory need some modification
- PR applies to NM & electromagnetic theory but NM need revision

Einstein accepted the last alternatives & laid to the foundation of SPECIAL RELATIVITY (SR) THEORY in 1905.

Postulates of special theory of Relativity

Einstein Principle of Relativity (EPR)

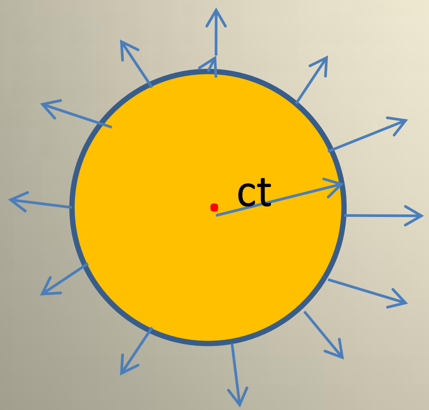
1. The fundamental laws of physics are same in all inertial frames. ie. PR.

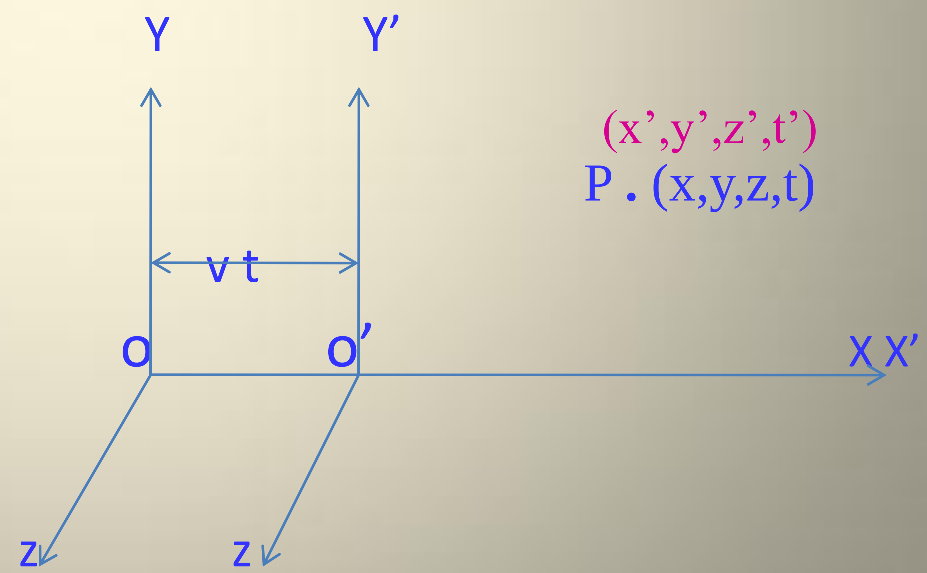
Constancy of speed of light

1. The speed of light in the free space is same for all inertial observer and is independent of the relative velocity of source of light and the observer.

Lorentz Transformation

- Consider two inertial frames S and S'
- S' is moving with uniform speed V w. r. to S.
- O & O' the observers at origins of S & S' observing the same event at any point p.





The point which is at rest w. r. to S' will move with velocity V relative to S in XX' direction. ie. The point $x' = 0$ is identical with $x = v\,t$ so that

$x' = \gamma\,(x - v\,t)$…………………………..(1)

Where γ is some function of v.

Since velocity of S' is along xx' direction , we have

$$y' = y \;\;\&\;\; z' = z \;……….(2)$$

Also, $\quad t' = \alpha\,t + \beta\,x \;…………..(3)$

where α & β are functions of v.

Equation of spherical wavefront is,

In S, $\quad x^2 + y^2 + z^2 = t^2 \;…………(4)$

In S' $\quad x'^2 + y'^2 + z'^2 = t'^2 \;………(5)$

Putting x', y', z', & t' in equation (5)

Equation (5) $\Rightarrow \gamma^2 (x - v\,t)^2 + y^2 + z^2 = c^2 (\alpha\, t + \beta\, x)^2$

ie. $\gamma^2 x^2 - 2x \gamma^2 v\, t + \gamma^2 v^2 t^2 + y^2 + z^2 = c^2 \alpha^2 t^2 + 2 \alpha \beta c^2 x\, t + \beta^2 c^2 x^2$

$(\gamma^2 - \beta^2 c^2) x^2 + y^2 + z^2 - 2xt (\gamma^2 v + \alpha \beta c^2) = (c^2 \alpha^2 - \gamma^2 v^2) t^2$ .....(6)

Equations (4) & (6) represents the same wavefront. On comparing their coefficients, we get,

$$\gamma^2 - \beta^2 c^2 = 1 \quad \text{..................(7)}$$

$$\gamma^2 v + \alpha \beta c^2 = 0 \quad \text{..................(8)}$$

$$c^2 \alpha^2 - \gamma^2 v^2 = c^2 \quad \text{..................(9)}$$

Eq. (8) – v . Eq.(7) $\Rightarrow$

$$\alpha \beta c^2 + \beta^2 c^2 v = - v$$

$$\alpha \beta c^2 + v (\beta^2 c^2 + 1) = 0 \quad \text{..........(10)}$$

Now Eq. (9) + v^2 . Eq. (7) $\Rightarrow$

$$c^2 \alpha^2 - \beta^2 c^2 v^2 = c^2 + v^2$$

i.e. $c^2 (\alpha^2 - 1) - v^2 (\beta^2 c^2 + 1) = 0$ ....(11)

Eq. (11) + v. Eq. (10) $\Rightarrow c^2 \alpha^2 - c^2 + \alpha \beta c^2 v = 0$

i.e. $1 - \alpha^2 = \alpha \beta v \Rightarrow \beta = \dfrac{1 - \alpha^2}{\alpha v}$

Putting this β in Eq. (10) we get,

$$c^2\alpha\left[\frac{1-\alpha^2}{\alpha v}\right] + v\left[c^2\left(\frac{1-\alpha^2}{\alpha v}\right)^2 + 1\right] = 0$$

$$c^2\frac{(1-\alpha^2)}{v} + c^2\left(\frac{1-\alpha^2}{\alpha^2 v}\right)^2 + v = 0$$

$$\text{ie. } c^2\alpha^2(1-\alpha^2) + c^2(1-\alpha^2)^2 + \alpha^2 v^2 = 0$$

$$\text{i.e. } c^2\alpha^2 - c^2\alpha^4 + c^2(1 - 2\alpha^2 + \alpha^4) + \alpha^2 v^2 = 0$$

$$c^2 - c^2\alpha^2 + \alpha^2 v^2 = 0 \Rightarrow c^2 = \alpha^2 (c^2 - v^2)$$

$$\alpha^2 = \frac{c^2}{c^2 - v^2} \Rightarrow \alpha^2 = \frac{1}{1 - \frac{v^2}{c^2}} \text{ or } \alpha = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Putting α^2 in Eq. (9) we get, $\left(\frac{c^2}{c^2 - v^2}\right)c^2 - \gamma^2 v^2 = c^2$

i.e. $c^4 - \gamma^2 v^2(c^2 - v^2) = c^2(c^2 - v^2)$

$\gamma^2 v^4 - \gamma^2 v^2 c^2 + c^2 v^2 = 0$

i.e. $\gamma^2(c^2 - v^2) = c^2 \Rightarrow \gamma^2 = \frac{c^2}{c^2 - v^2} \Rightarrow \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

$$\therefore \alpha = \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Now, $\beta = \frac{1 - \alpha^2}{\alpha v} = \frac{1 - \frac{c^2}{c^2 - v^2}}{\alpha v} = \frac{c^2 - v^2 - c^2}{\alpha v(c^2 - v^2)}$

$$\therefore \beta = \frac{-v^2}{\alpha v(c^2 - v^2)} = \frac{-v}{(c^2 - v^2)\alpha} = \frac{-v}{(c^2 - v^2)}\frac{\alpha}{\alpha^2}$$

$$= \frac{-v}{(c^2 - v^2)} \frac{\alpha}{\frac{c^2}{(c^2 - v^2)}} = -\frac{\alpha v}{c^2}$$

We have $t' = \alpha t + \beta x$ $\quad \therefore \quad t' = \alpha t - \frac{\alpha v}{c^2} x = \alpha\left(t - \frac{v x}{c^2} \right)$

$\therefore$ The Lorentz transformation equations are

$x' = \alpha (x - v t)$

$y' = y$

$z' = z$

$t' = \alpha\left(t - \frac{v x}{c^2} \right)$ where $\alpha = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

The quantity $\alpha = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ is called as Lorentz contraction factor

Lorentz Inverse Transformations

Lorentz Inverse Transformations are,

$$x = \alpha (x' + v t)$$

$$y = y'$$

$$z = z'$$

$$t = \alpha(t' + \frac{v x}{c^2}) \qquad \text{where } \alpha = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Ex. Show that for low value of v the LT reduces to GT

Sol^n: The Lorentz transformation equations are

$$x' = \alpha (x - v t)$$

$$y' = y$$

$$z' = z$$

$$t' = \alpha(t - \frac{v x}{c^2}) \quad \text{where } \alpha = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

If v < < c then v/c < < 1 $\therefore \frac{v^2}{c^2} \to 0 \;\&\; \alpha = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = 1$

$\therefore$ LT reduces to ,

$$x' = x - v t$$

$$y' = y$$

$$z' = z$$

$t' = t$ which are GT.

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THANKS !